Figure 4 shows the dependence of the size of the first pressure maximum on the gas volume in the dimensionless variables $P_m = P_m(\alpha_V)$, $P_m = P_{max}/P_1$, where P_{max} is the pressure in the first peak. The initial increase in pressure with an increase in α_V – due to intensification of the effect of the gas cavity on the acceleration of the liquid – is subsequently replaced by a smooth decrease in α_V due to the increasing role of hydraulic losses [4]. At $\alpha_V > 0.007$, the experimental data is somewhat higher than the theoretical results. This may be connected with the fact that, in the calculations, hydraulic losses were accounted for by means of constant coefficients ξ_1 and ξ_2 taken for a steady flow. For nonsteady motion, however, hydraulic resistance may depend on the instantaneous values of fluid velocity and acceleration.

Thus, in the investigated case of relatively slow loading of a hydraulic system, the presence of a small volume of gas ($\alpha_V \sim 0.01$) leads to a substantial (albeit less than for instantaneous loading) increase in the pressures realized in the transient. The possibly dangerous effect of a localized gas volume such as that examined here should be considered in the analysis of transients in hydropneumatic systems.

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WAVE FLOWS OF A CONDUCTING VISCOUS FLUID FILM IN A TRANSVERSE MAGNETIC FIELD

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Investigation of the wave regimes occurring in thin layers of a viscous weak-conducting fluid in magnetic and electrical fields is of interest in connection with the prospective utilization of film flows in nuclear power [1] and other technological processes. Experimental and theoretical investigations of wave effects in structures that occur on the free surface of an ordinary (non-electrically conducting) viscous fluid showed that these phenomena influence the stability and evolution of the film flows substantially [2-4]. The theory of the wave motion of a laminar viscous film surface was first developed by Kapitsa [2]. The critical value of the Reynolds number was obtained for which a wave mode is built up in the film when it is exceeded. It is shown that the mass transfer is improved in films in the wave mode as compared with ordinary flow conditions. At this time magnetohydrodynamic flows of conducting viscous fluid films are studied intensively [5-7]. A mathematical model is proposed in [5] for a flow with a free surface of the liquid-metal diaphragm of a power plant. The asymptotic of the surface of the spreading film in transverse electrical and magnetic fields is presented in [6]. The stability of a laminar flow of an electrically conducting fluid film is considered in an induction-free approximation in [7] on the basis of the Orr–Sommerfeld equation.

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An equation is found in this paper that describes the longwave oscillations of the surface of this conducting fluid layer in crossed electrical and magnetic fields. Conditions for which different limit case of this equation hold are analyzed.

1. FORMULATION OF THE PROBLEM

Viscous fluid flow in external stationary magnetic $\mathbf{H}_0$ and electrical $\mathbf{E}_0$ fields can be described by the equations of magnetohydrodynamics, which reduce to the following for small magnetic Reynolds Re numbers (induction-free approximation) [8]:

$$\partial \mathbf{u}/\partial t + (\mathbf{u}\nabla)\mathbf{u} = -\nabla P/\rho_0 + \nu\Delta\mathbf{u} + \mathbf{F}_c,\ \operatorname{div}\mathbf{u} = 0, \tag{1.1}$$

$$\mathbf{F}_c = (\rho_0 c)^{-1}[\mathbf{j}\times\mathbf{H}_0] + \mathbf{g},\ \mathbf{j} = \sigma(\mathbf{E} + c^{-1}[\mathbf{u}\times\mathbf{H}_0]),\ \mathbf{E} = -\operatorname{grad}\varphi;$$
$$\Delta\varphi = c^{-1}\mathbf{H}_0\cdot\operatorname{rot}\mathbf{u}. \tag{1.2}$$

Here u is the fluid velocity, P, ρ_0 its pressure and density, c is the speed of sound, ν is the kinematic viscosity, σ the fluid conductivity, and g the acceleration of gravity.

Let us consider a plane flow: u = {u, v, 0}, $\partial u/\partial z = 0$. Equation (1.2) for the potential here goes over into the Laplace equation $\Delta\varphi = 0$, from which $\mathbf{E} = \mathbf{E}_0$ (constant external field).

Therefore, for $\mathrm{Re}_m \ll 1$ we have for the flow of a conductive fluid in the electromagnetic fields displayed in Fig. 1

$$\mathbf{H} = \mathbf{H}_0,\ \mathbf{E} = \mathbf{E}_0. \tag{1.3}$$

Conditions on the interfacial boundary between the fluid and the solid surface S_s and on the free surface S_p [9] must be appended to the system (1.1) (F(r, t) = 0 is the equation of the free surface)

$$\mathbf{u}(\mathbf{r}, t) = 0, \mathbf{r}\in S_s, [P - P_a + \sigma_*(R_1^{-1} + R_2^{-1})]\,n_i = \sigma_{ik}n_k,$$
$$F_t + (\mathbf{u}\nabla)F = 0, \tag{1.4}$$

where P_a is the atmospheric pressure, σ_* is the coefficient of surface tension, R_1, R_2 are the principal radii of curvature at the point r of the surface S_p; n_i are the cosines of the normal to the free surface and σ_{ik} is the viscous stress tensor.

Let us consider the plane flow of a liquid film of thickness y = h(x, t) in constant transverse electrical and magnetic fields (see Fig. 1). Let us introduce dimensionless variables and parameters

$$x' = x/l,\ y' = y/h_0,\ h' = h/h_0,\ t' = t/t_0,$$
$$P' = (P - P_a)/P,\ u' = u/u_0,\ v' = v/v_0,\ v_0 = \delta u_0,$$
$$u_0 = \sqrt{gh_0},\ \delta = h_0/l = u_0^2/(gl) = \mathrm{Fr},\ P_0 = \rho_0 u_0^2,\ \alpha = cE_0/u_0H_0 \tag{1.5}$$

(Fr is the Froude number, l is the characteristic length of the perturbation on the film surface, and h_0 is the mean film thickness). We later omit the primes on the dimensionless variables by setting x = x', y = y', h = h', etc. The system (1.1), (1.2), and (1.3) with boundary conditions (1.4) is here written in dimensionless variables (1.5) as

$$u_t + uu_x + vu_y + P_x = (1/\mathrm{Re})(u_{xx} + u_{yy}/\delta^2) + \mathrm{Ha}^2(\alpha - u)/(\delta^2\mathrm{Re}); \tag{1.6}$$

$$\delta^2(v_t + uv_x + vv_y) + P_y = (\delta^2/\mathrm{Re})(v_{xx} + v_{yy}/\delta^2) - 1; \tag{1.7}$$

$$u_x + v_y = 0; \tag{1.8}$$

$$u = v = 0,\ y = 0; \tag{1.9}$$

$$P = \frac{2\left[v_y + \delta^2\eta_x^2 u_x - \eta_x\left(u_y + \delta^2 v_x\right)\right]}{\mathrm{Re}\left(1 + \delta^2\eta_x^2\right)} - \mathrm{We}\frac{\eta_{xx}}{(1 + \delta^2\eta_x^2)^{3/2}},\ y = 1 + \eta; \tag{1.10}$$

$$(u_y + \delta^2 v_x)(1 - \delta^2\eta_x^2) - 2\delta^2\eta_x(u_x - v_y) = 0,\ y = 1 + \eta; \tag{1.11}$$

$$\eta_t + u\eta_x = v \tag{1.12}$$

($\mathrm{Re} = u_0 l/\nu$, $\mathrm{Ha} = H_0 h_0\sqrt{\sigma/\rho_0\nu}/c$, $\mathrm{We} = \sigma_*/(g\rho_0 l)^2$ are the Reynolds, Hartman, and Weber numbers).

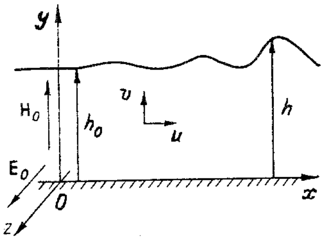

Fig. 1

2. LONGWAVE APPROXIMATION

Let us find the solution of the problem (1.6)-(1.12) in the form of an expansion in the small parameter $\delta \ll 1$ (longwave approximation)

$$\begin{gathered} \eta = \delta\eta^{(1)} + \delta^2\eta^{(2)} + \ldots,\quad P = P^{(0)}(y) + \delta P^{(1)} + \delta^2 P^{(2)} + \ldots, \\ v = -\delta\partial\psi^{(1)}/\partial x - \delta^2\partial\psi^{(2)}/\partial x - \ldots, \\ u = u^{(0)}(y) + \delta\partial\psi^{(1)}/\partial y + \delta^2\partial\psi^{(2)}/\partial y + \ldots, \end{gathered} \tag{2.1}$$

where $\psi = \delta\psi^{(1)} + \delta^2\psi^{(2)} + \ldots$ is the stream function and $u^{(0)}(y)$, $P^{(0)}(y)$ are solutions of (1.6)-(1.8) corresponding to the stationary fluid flow ($\eta^{(0)} = v^{(0)} = 0$). The equations for $P^{(0)}$ and $u^{(0)}$ are

$$u^{(0)}_{yy} + \mathrm{Ha}^2\left(\alpha - u^{(0)}\right) = 0.\ P^{(0)}_y = -1,\ u^{(0)}_y(1) = u^{(0)}(0) = 0,\ P^{(0)}(1) = 0. \tag{2.2}$$

Integrating (2.2) we obtain

$$u^{(0)} = \alpha\left[1 - \frac{\operatorname{ch}\{\mathrm{Ha}(y-1)\}}{\operatorname{ch}\mathrm{Ha}}\right],\ P^{(0)} = 1 - y. \tag{2.3}$$

Let us introduce the new variables ξ and τ in place of x and t: $\tau = \delta t$, $\xi = x - Ft$. Substituting (2.1) and (2.3) and the expressions for τ and ξ into (1.6)-(1.8) and omitting components of order δ^2 and higher, we have

$$\psi^{(1)}_{yyy} - \mathrm{Ha}^2\,\psi^{(1)}_y = 0,\ P^{(1)}_y = -\mathrm{Re}^{-1}\,\psi^{(1)}_{\xi yy}. \tag{2.4}$$

The boundary conditions for y = 0 and on the free surface, referred to y = 1 are written in this approximation in the form

$$y = 0,\ \psi^{(1)} = \psi^{(1)}_y = 0; \tag{2.5}$$

$$\begin{gathered} y = 1,\ P^{(1)} = \eta^{(1)} - 2\mathrm{Re}^{-1}\,\psi^{(1)}_{\xi y} - \mathrm{We}\,\eta^{(1)}_{\xi\xi}, \\ y = 1,\ u^{(0)}_{yy}\eta^{(1)} + \psi^{(1)}_{yy} = 0,\ \left(u^{(0)} - F\right)\eta^{(1)}_\xi + \psi^{(1)}_\xi = 0. \end{gathered} \tag{2.6}$$

We find the solution of the system (2.4) with the boundary conditions (2.5) and (2.6) as follows for:

$$\begin{gathered} \psi^{(1)} = \frac{\alpha(\operatorname{ch}\mathrm{Ha}\,y - 1)}{\operatorname{ch}^2\mathrm{Ha}}\,\eta^{(1)}, \\ P^{(1)} = -\mathrm{We}\,\eta^{(1)}_{\xi\xi} - \alpha\frac{\mathrm{Ha}}{\mathrm{Re}}\frac{(\operatorname{sh}\mathrm{Ha}\,y + \operatorname{sh}\mathrm{Ha})}{\operatorname{ch}^2\mathrm{Ha}}\,\eta^{(1)}_\xi + \eta^{(1)}. \end{gathered} \tag{2.7}$$

To determine $\eta^{(1)}$ we use equations of second approximation in δ. Substituting (2.1), (2.3), (2.7) into (1.6), (1.9)-(1.12) and omitting components of order δ^3 and higher, we obtain

$$\psi^{(2)}_{yyy} - \mathrm{Ha}^2\,\psi^{(2)}_y = -\psi^{(1)}_{\xi\xi y} + \mathrm{Re}\left[P^{(1)}_\xi - u^{(0)}_y\psi^{(1)}_\xi + u^{(0)}\psi^{(1)}_{\xi y} - F\psi^{(1)}_{\xi y}\right]; \tag{2.8}$$

$$y = 0,\ \psi^{(2)} = \psi^{(2)}_y = 0; \tag{2.9}$$

$$\begin{gathered} y = 1,\ \eta^{(2)}u^{(0)}_{yy} + \eta^{(1)}\psi^{(1)}_{yyy} + \psi^{(2)}_{yy} - \psi^{(1)}_{\xi\xi} = 0, \\ y = 1,\ \eta^{(1)}_\tau + \left(u^{(0)} - F\right)\eta^{(2)}_\xi + \left(\eta^{(1)}\psi^{(1)}_\xi\right)_\xi + \psi^{(2)}_\xi = 0. \end{gathered} \tag{2.10}$$

Taking account of (2.3) and (2.7), we find a closed equation for $w = \mathrm{Re}\,\eta^{(1)}$ ($s = \tau 2\,\mathrm{Re}\,\alpha\mathrm{Ha}\ \tanh\mathrm{Ha}/\cosh^2\mathrm{Ha}$) from (2.8)-(2.10).

$$w_s + ww_\xi + \varepsilon w_{\xi\xi} + \beta w_{\xi\xi\xi} + \omega w_{\xi\xi\xi\xi} = 0;$$
$$\varepsilon = [\alpha \mathrm{Ha}(3\mathrm{Ha} - 3 \operatorname{th} \mathrm{Ha} - \mathrm{Ha} \operatorname{th}^2 \mathrm{Ha}) - 2\alpha^{-1}(\mathrm{Ha} \operatorname{cth} \mathrm{Ha} - 1)\operatorname{ch}^2 \mathrm{Ha}]/(4\mathrm{Ha}^4);$$
$$\beta = (1 - \operatorname{ch} \mathrm{Ha})(2 - 2 \operatorname{ch} \mathrm{Ha} - \mathrm{Ha} \operatorname{sh} \mathrm{Ha})/(2\mathrm{Re}\mathrm{Ha}^3), \tag{2.11}$$
$$\omega = \mathrm{We} \operatorname{ch}^2 \mathrm{Ha}(\mathrm{Ha} \operatorname{cth} \mathrm{Ha} - 1)/(2\alpha \mathrm{Ha}^4).$$

Equation (2.11) is an evolutionary equation with nonlinearity of Burgers and KdV type. The interpretation of w as the momentum flux density and $w^2/2$ as the energy flux density is standard for such equations [10]. Momentum and energy conservation laws follow from (2.11) for the localized perturbations: $\frac{\partial}{\partial s}\int_{-\infty}^{\infty} w\, d\xi = 0, \frac{\partial}{\partial s}\int_{-\infty}^{\infty} (w^2/2)\, d\xi = \varepsilon \int_{-\infty}^{\infty} w_\xi^2\, d\xi - \omega \int_{-\infty}^{\infty} w_{\xi\xi}^2\, d\xi.$

This means that the component $\varepsilon w_{\xi\xi}$ in (2.11) corresponds to energy pumping into the wave while $\omega w_{\xi\xi\xi\xi}$ to its dissipation. Pumping is realized at low frequencies and its dissipation at high frequencies. The nonlinear component ww_ξ affords the possibility of energy being pumped from oscillations at low frequency to high-frequency oscillations. The term $\beta w_{\xi\xi\xi}$ in (2.11) describes energy dispersion.

It is easy to show that for any Ha > 0 the coefficients are $\omega > 0$ and $\beta > 0$. The sign of ε is determined by the quantity α: if $\alpha^2 > (\alpha^*)^2 = 2(\mathrm{Ha} \coth \mathrm{Ha} - 1)\cosh^2\mathrm{Ha}/[\mathrm{Ha}(3\mathrm{Ha} - 3 \tanh \mathrm{Ha} - \mathrm{Ha} \tanh^2\mathrm{Ha}] > 0$, then $\varepsilon > 0$; while for $\alpha < \alpha^*$, $\varepsilon < 0$. It hence follows that energy pumping into the wave ($\varepsilon > 0$) holds if the electrical field $\mathbf{E}_0$ is sufficiently large. In this connection, an analogy can be noted between the process under consideration and the runoff of a viscous, non-electrically conductive fluid on an inclined plane [4]. A critical parameter also exists here, the angle φ_0 between the inclined plane and the vertical. If $\varphi < \varphi_0$, then the oscillations on the fluid surface damp out, if $\varphi > \varphi_0$, the perturbation energy does not decrease.

3. PERIODIC SOLUTIONS OF (2.11)

Let us consider the solutions of (2.11) of the stationary travelling wave type in which w is a function of the variable $\theta = \xi - Ds$ (D is the velocity of wave propagation). In this case (2.11) is written in the form

$$\omega w^{\mathrm{IV}} + \beta w''' + \varepsilon w'' + \omega w' - w' = 0. \tag{3.1}$$

which allows solutions of wave front, solitary, and periodic wave type. Integrating (3.1) with respect to θ, we obtain

$$\omega H''' + \beta H'' + \varepsilon H' + H(2H - D) = 0, \quad H = (w - D)/4 + \sqrt{(D/4)^2 + q/8} \tag{3.2}$$

(q is the constant of integration). For periodic waves we find q from the condition $\int_0^\lambda w d\theta = 0$ (λ is the wavelength), which is a consequence of defining w as the deviation from the mean film thickness.

Equation (3.2) has two homogeneous stationary solutions: $H = H_1 = 0$ and $H = H_2 = D/2$. The differential equation (3.2) is written in the form of a system of first-order equations

$$dH/d\theta = Q,\ dQ/d\theta = R,\ \omega dR/d\theta = -\beta R - \varepsilon Q - H(2H - D). \tag{3.3}$$

To investigate (3.3) we use methods of the theory of dynamical systems and θ is interpreted as the time ($-\infty < \theta < \infty$, $\theta \to \infty$) [3]. The dynamical system (3.3) has two fixed points $S_1(0, 0, 0)$, and $S_2(D/2, 0, 0)$, corresponding to the stationary solutions H_1 and H_2. The eigenvalues of the Jacobi matrix of the linearized system (3.3) are determined in the neighborhood of the fixed point S_k from

$$\omega\rho^3 + \beta\rho^2 + \varepsilon\rho + (-1)^k D = 0.$$

For $|D| < \beta\varepsilon/\omega$ the real parts of the roots of the characteristic equation are strictly negative, and therefore, the fixed points S_k (k = 1, 2) are stable. For k = 1, $D = -\beta\varepsilon/\omega$ and

$k = 2$, $D = \beta\varepsilon/\omega$ the characteristic equation has the roots $\rho_1 = -\beta/\omega$ and $\rho_{2,3} = \pm i\sqrt{\varepsilon/\omega}$. If $D > \beta\varepsilon/\omega$, then the fixed point S_2 becomes linearly unstable, while S_1 is linearly unstable for $D < -\beta\varepsilon/\omega$, i.e., bifurcation occurs for generation of the cycle for $D = \pm\beta\varepsilon/\omega$.

Let us find the solution of the system (3.3) and therefore of (3.2) also near the bifurcation point $D = \beta\varepsilon/\omega$. Let us use the algorithm proposed in [9] that is based on application of the theorem of a central manifold and on reduction of an autonomous system corresponding to (3.2) to normal Poincaré form. For $0 < D - D_* < \delta \ll 1$ such a solution has the form ($\gamma = \beta/\sqrt{\varepsilon\omega}$, $D_* = \beta\varepsilon/\omega$)

$$H = \frac{D}{2} + \frac{\delta}{1+\gamma^2}\left[\cos\frac{2\pi\theta}{T} - \gamma\sin\frac{2\pi\theta}{T}\right] + \frac{1}{3(4+\gamma^2)}\left(\frac{\delta}{1+\gamma^2}\right)^2\left[\gamma(5-\gamma^2)\times\right.$$
$$\left.\times\cos\frac{4\pi\theta}{T} + 2(1-2\gamma^2)\sin\frac{4\pi\theta}{T} - 3(\gamma^2+4)(1+\gamma^2)/\gamma\right] + O(\delta^3), \tag{3.4}$$

where $T = 2\pi\sqrt{\omega/\alpha}\{1 + 2\delta^2(4\gamma^2 + 25)/[3(\gamma^2 + 4)(\gamma^2 + 1)^2]\}$ is the period and $\delta^2 = (D - D_*)/\mu_2 + O[(D - D_*)^2]$; $\mu_2 = 2(\gamma^2 + 8)/[\gamma(\gamma^2 + 4)(\gamma^2 + 1)]$. Since the Floquet index is $\beta = \beta_2\delta^2 + O(\delta^4)$; $\beta_2 = -2(\gamma^2 + 8)/[\gamma(\gamma^2 + 4)(\gamma^2 + 1)]$, then (3.4) is stable by virtue of the Hopf theorem [9]. For certain $|D| > \beta\varepsilon/\omega$, Eq. (3.2), and therefore also (2.11) have solutions of solitary wave type. In certain particular cases they can be represented analytically [11], although they can generally only be found numerically.

Therefore, an equation is obtained that describes perturbation wave propagation over a conducting fluid film surface in transverse constant magnetic and electrical fields. The equation is a generalization of the known Burgers–Korteweg–deVries equation. Physical conditions are indicated for which different limit forms of the equation are valid. It is shown that this equation has two bifurcation points for generation of a cycle, and consequently, periodic solutions near these points.

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